

HILBERT STABILITY OF RANK-TWO BUNDLES ON CURVES

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1. Let k be an algebraically closed field, and let d and g be two integers with $g \geq 2$ and $d \geq 1000g(g-1)$. Let $n = d + 2 - 2g$, and let W be a vector space of dimension n . G will denote the grassmannian of all codimension-two subspaces of W , and \mathcal{E} will denote the universal rank-two bundle on G . In this paper, a curve will be a connected one-dimensional projective scheme. Let C be a curve on G , i.e., C is a subscheme of G which is a curve, and consider $E = \mathcal{E}_C = \mathcal{E}|_C$. Let $P_C(m) = \chi((\det E)^{\otimes m})$ be the Hilbert polynomial of C where $\det E = \wedge^2 E$. We let $S_{g,d}$ be the set of all curves C on G with $P_C(m) = dm + 2 - 2g$. Thus $S_{g,d}$ is the set of all curves of genus g and degree d on G .

Now W is identified with $H^0(G, \mathcal{E})$, so given $C \in S_{g,d}$, there is a natural map

$$\varphi_1: W \rightarrow H^0(C, E).$$

We will identify W with $H^0(C, E)$ if φ_1 is an isomorphism. Thus we obtain a map

$$\varphi_2: \wedge^2 W \rightarrow H^0(C, \wedge^2 E).$$

So for any positive integer m , we obtain a map

$$\varphi_3: S^m(\wedge^2 W) \rightarrow H^0(C, (\det E)^{\otimes m}).$$

We may and do choose m so that φ_3 is onto, so that $h^0(C, (\det E)^{\otimes m}) = P_C(m)$ for any $C \in S_{g,d}$. Thus we finally obtain a map

$$\varphi_C^m: \wedge^{P_C(m)} S^m(\wedge^2 W) \rightarrow \wedge^{P_C(m)} H^0(C, (\det E)^{\otimes m}) \cong k.$$

We say $C \subseteq G$ is m -Hilbert stable (resp., m -Hilbert semistable) if φ_C^m is properly stable (resp., semistable) under the induced action of $SL(W)$ in the

Received January 10, 1983, and, in revised form, July 7, 1983. Research partially supported by NSF Grant MCS79-03171.

terminology of Mumford, i.e., φ_C^m has closed orbit and finite stabilizer (resp., 0 is not in the closure of the orbit of φ_C^m). We say C is Hilbert stable if it is m -Hilbert stable for $m \gg 0$. We say a pair (C, E) consisting of a curve C and vector bundle E of rank two is m -Hilbert stable if (C, E) occurs as an m -Hilbert stable curve in $S_{g,d}$.

Now if E is a rank-two bundle on a smooth curve C , and L is a subbundle of E of maximal degree, we define $l_E = \deg E - 2 \deg L$. Recall that E is stable if $l_E > 0$ and semistable if $l_E \geq 0$.

A curve C is nodal if C is reduced and has only nodes as singularities. Let ω_C denote the dualizing sheaf of such a curve. Recall C is stable (resp., semistable) if ω_C has positive degree (resp., nonnegative degree) on each component of C [5]. For each semistable curve, the sections of $\omega_C^{\otimes 3}$ define a map to P^{5g-5} , and the image of C is a stable curve denoted C_s . C_s is obtained from C by collapsing all components on which ω_C is trivial. These components are smooth rational curves meeting the rest of C in exactly two points. A semistable subcurve C' of C is a subcurve which is the inverse image of a node of C_s .

We fix g for the rest of the paper.

Theorem 1.1. *There is a D so that for each $d \geq D$, there is an M depending on d so that if $m \geq M$, and C is a smooth curve in $S_{g,d}$ with $W = H^0(C, E)$, then C is m -Hilbert stable (resp., semistable) if and only if \mathcal{E}_C is stable (resp., semistable).*

Theorem 1.2. *For g and d given, there is an M so that if $m \geq M$ and $C \in S_{g,d}$ is m -Hilbert semistable, then C is semistable as a curve and $W = H^0(C, \mathcal{E}_C)$.*

The proof of Theorem 1.1 is given in §§2–5 and that of Theorem 1.2 in §§6–9.

Now in §10 we will suppose $C \in S_{g,d}$ is m -Hilbert stable for m sufficiently large, and study $E = \mathcal{E}_C$. First we will show that if Q is a quotient line bundle of E , then

$$(1.3.1) \quad \deg E \leq 2 \deg Q.$$

Now let C' be a semistable subcurve of C . E is said to be acceptable on C' if either

(1.3.2.1) C' has one component and so is isomorphic to \mathbf{P}^1 , and $E_{C'}$ is $\mathcal{O} \oplus \mathcal{O}(1)$ or $\mathcal{O}(1) \oplus \mathcal{O}(1)$ or

(1.3.2.2) C' has two components C_1 and C_2 , and $E_{C'}$ is isomorphic to $\mathcal{O} \oplus \mathcal{O}(1)$. Further, $E_{C'}$ has no quotient isomorphic to $\mathcal{O}_{C'}$.

We will show

(1.3.3) E is acceptable on each semistable subcurve of C .

Finally, let d be odd and suppose C_s is an irreducible curve with a node. Let \tilde{C} be the normalization of C_s . Then \tilde{C} maps to C as a component of C if $C \neq C_s$. Thus we may consider \tilde{E} , the pullback of E to \tilde{C} . Then we will show

(1.3.4) If $C = C_s$ and d is odd, then $l_{\tilde{E}} \geq -1$. If $C \neq C_s$ then \tilde{E} is semistable.

We wish to thank Ed Griffin for pointing out an error in an earlier version of this paper.

2. Let C be a curve in $S_{g,d}$. We wish to apply the Hilbert-Mumford numerical criterion to φ_C^m . First, a weighted basis (X_i, r_i) of W is an ordered basis of W together with rational numbers r_i with $r_1 \geq r_2 \geq \dots \geq r_n$. If the r_i are integers, and their sum is zero, we call B standard. A standard weighted basis determines a one-parameter subgroup of $SL(W)$ via

$$X_i^{\lambda(\alpha)} = \alpha^{r_i} X_i.$$

Every $1 - PS$ occurs in this way. A weighted basis B of W gives rise to weighted bases on the representations of $SL(W)$ discussed above, as shown in the table.

REPRESENTATION	BASIS ELEMENT	WEIGHT
$\wedge^2 W$	$Y_I = X_{i_1} \wedge X_{i_2}$	$r_I = r_{i_1} + r_{i_2}$
$S^m \wedge^2 W$	$M_g = Y_{I_1} \cdots Y_{I_m}$	$r_g = \sum_{k=1}^m r_{I_k}$
$\wedge^{P(m)} S^m \wedge^2 W$	$M_{g_1} \wedge \cdots \wedge M_{g_{P(m)}}$	$\sum_{k=1}^{P(m)} r_{g_k}$

If B is standard, so is each of these bases, and each diagonalizes the action of λ_B on the corresponding representation. The coordinate corresponding to $M_1 \wedge \cdots \wedge M_{P(m)}$ does not vanish at φ_C^m if and only if the images under φ_C^m of $M_1, \dots, M_{P(m)}$ in $H^0(C, \wedge^2 E^{\otimes m})$ form a basis there. We will call such a basis a B -base of $H^0(C, \wedge^2 E^{\otimes m})$, and denote by $w_B(m)$ or $w_B(m, C)$ the minimum weight of such a basis. Each B determines a weighted filtration $F_B = \{(V_i, r_i)\}$ on W by $V_i = \text{span}\{X_i, \dots, X_n\}$. A useful observation is

Lemma 2.1. *If $F_B = F_{B'}$, then $w_B(m) = w_{B'}(m)$.*

Recall the Hilbert-Mumford numerical criterion: a point x of a representation V of a reductive algebraic group G has stable orbit if and only if, given any nontrivial $1 - PS$ λ of G and coordinates which diagonalize the action of λ on V , there is a coordinate not vanishing at x whose λ -weight is negative. The

discussion above therefore gives

Theorem 2.2. (C, E) is m -Hilbert stable (resp., semistable) if and only if for any nontrivial standard weighted basis B of W , $w_B(m) < 0$ (resp., $w_B(m) \leq 0$).

Corollary 2.3. (C, E) is m -Hilbert stable (resp., semistable) if for any nontrivial weighted basis B of W

$$w_B(m) < (\text{resp.}, \leq) \frac{2mh^0(C, (\wedge^2 E)^{\otimes m})}{h^0(C, E)} \sum_{i=1}^n r_i.$$

Proof. Since both sides of the inequality are linear in the r_i jointly, it suffices to prove this when the r_i are integers. We then associate to B the standard weighted basis $B' = \{(X_i, s_i)\}$, where $s_i = nr_i - \sum_{j=1}^n r_j$. The B' -weight of a monomial of degree m in the exterior products $X_i \wedge X_j$ equals n times its B -weight minus $2m \sum_{j=1}^n r_j$. Since any B -basis contains $h^0(C, (\wedge^2 E)^{\otimes m})$ elements,

$$w'_B(m) = h^0(C, E)w_B(m) - 2mh^0(C, (\wedge^2 E)^{\otimes m}) \sum_{i=1}^n r_i.$$

The corollary now follows immediately from Theorem 2.2.

We will say C is m -stable with respect to a weighted basis B if the inequality of Corollary 2.3 holds for $w_B(m)$. From the linearity of this inequality in the $\{r_i\}$ jointly, we see that we are free to translate and rescale the weights so that $r_1 \geq r_2 \geq \dots \geq r_n = 0$ and $\sum_{i=1}^n r_i = 1$. We say a weighted basis B satisfying these conditions is normalized. Note also that if the r_i are integers, then each side of the inequality in Corollary 2.3 is represented for large m by a polynomial of degree two in m whose leading term is of the form $\frac{1}{2}em^2$ with e an integer (cf. [6]). We call e the normalized leading coefficient, written n.l.c., of this polynomial, and define e when the r_i are rational using the linearity of e in the r_i jointly.

Corollary 2.4. Fix g, d and a real number $\epsilon > 0$. Then we can choose an integer M (depending only on g, d and ϵ) so that the statement below is verified:

If B is a normalized weighted basis of W and

$$\text{n.l.c. } w_B(m, C) \leq \frac{4d}{n} - \epsilon r_1,$$

$C \in \mathcal{S}_{g,d}$, then for all $m \geq M$, C is m -stable with respect to B .

Proof. This can be established by techniques similar to the proof of Proposition 1.2 of [1].

Now if L is a subbundle of E with degree $\frac{1}{2} \deg E$ and $W \cong H^0(C, E)$, we can consider the normalized basis which assigns weight 0 to every element of $H^0(C, L)$ and equal weight to every element of $W/H^0(L)$. such a weighted

basis will be said to be special for C . In this situation, we have

Proposition 2.5. (i) *There is a D so that for each $d \geq D$, there is an $\varepsilon > 0$ so that if $C \in S_{g,d}$ is smooth with $W = H^0(C, E)$ and B is a normalized weighted basis of W which is not special for C , then*

$$\text{n.l.c. } w_B(m, C) \leq \frac{4d}{n} - \varepsilon(r_1 - r_n).$$

(ii) *There is an M so that if $m \geq M$ and B is a normalized special basis of $W = H^0(C, E)$, then*

$$w_B(m) = \frac{2mh^0(C, (\wedge^2 E)^{\otimes m})}{h^0(C, E)}.$$

Actually in (i) we will fix $C \in S_{g,d}$ and B , and show

$$\text{n.l.c. } w_B(m) < \frac{4d}{d + 1 - g},$$

and leave the question of the uniformity of ε with respect to C, E and B to the reader.

This is the key step to Theorem 1.1. The proof occupies the next three sections:

3. For §§3, 4 and 5 we fix a smooth curve C of genus g and a vector bundle E on C . Let $l_E = d - 2d_L$ where L is a linesubbundle of E of maximal degree. If E is decomposable, $l_E \leq 0$ but can be arbitrarily negative. However

Proposition 3.1 (Nagata [7]). *If E is indecomposable, $2 - 2g \leq l_E \leq g$.*

If L is a sublinebundle of E , we let $M_L = E/L$ and write M for M_L if the context determines L . We say L is nice if both L and M both have degree at least $2g + 1$.

Lemma 3.2. *If L is a nice subbundle of an indecomposable E , and U is any complement to $H^0(C, L)$ in $H^0(C, E)$, then the following hold:*

- (i) *The projection from E to M maps U isomorphically onto $H^0(C, M)$.*
- (ii) *E is generated by $H^0(C, L)$ and U .*
- (iii) *The map $\phi_{L,M}: H^0(C, L) \otimes H^0(C, M) \rightarrow H^0(C, L \otimes M)$ is surjective.*
- (iv) *The map ϕ_2 takes $H^0(C, L) \wedge U$ onto $H^0(C, \wedge^2 E)$.*

Moreover if $\text{deg } E \geq \max(5g + 1, 4g + 2 - l_E)$, and E indecomposable, then E has a nice linesubbundle.

Proof. For the last statement, note that since $\frac{1}{2}(\text{deg } E - g) \geq 2g + 1$ and $l_E \leq g$, E must have a sublinebundle L of degree at least $2g + 1$. The quotient M_L has degree $\text{deg } E - \text{deg } L \geq \frac{1}{2}(\text{deg } E + l_E) \geq 2g + 1$.

The long exact sequence associated to the composition series $0 \rightarrow L \rightarrow E \rightarrow M \rightarrow 0$ is $0 \rightarrow H^0(C, L) \rightarrow H^0(C, E) \rightarrow H^0(C, M) \rightarrow 0$ by the hypothesis on

L and M , which gives (i). If $P \in C$, let S be a section of L not vanishing at P , and let \tilde{T} be a section in U whose image in $H^0(C, M)$ is nonzero at P . Then S and \tilde{T} generate E at P , which gives (ii). Since L and M have degree at least $2g + 1$, the surjectivity of $\phi_{L,M}$ follows from [5, Theorem 6, p. 52]. Now observe that $L \otimes M = \wedge^2 E$ and that if $S \in H^0(C, L)$, $T \in H^0(C, M)$ and \tilde{T} is the section in U lying over T , then $\phi_2(S \wedge \tilde{T}) = \phi_{L,M}(S \otimes T)$; this yields (iv).

Now for §§3, 4 and 5, we suppose E is semistable and $W = H^0(C, E)$. We next recall a Proposition (3.2) which follows from results of [4] concerning stability of line bundles on C . While we will use some results on multiplicities to obtain Proposition 3.2, they do not appear in its statement and will not be used elsewhere. For definitions and a discussion of these multiplicities see [4]. Let $S = \{(S_i, \sigma_i)\}$ be a weighted basis of $H^0(C, L)$ where L is a very ample line bundle on C . Then for large m , $S^m H^0(C, L)$ maps onto $H^0(C, L^{\otimes m})$, and we define $w_S(m)$ to be the least weight of a basis of $H^0(C, L^{\otimes m})$ consisting of monomials in the S_i . We let \tilde{L} be the pullback of L to $C \times \mathbf{A}^1$. If the σ_i are nonnegative integers decreasing to zero, we define an ideal sheaf \mathcal{G}_S on $C \times \mathbf{A}^1$ by $\Gamma(\mathcal{G}_S \cdot \tilde{L}) = \langle S_i t^{\sigma_i} \rangle$, where t is a parameter on \mathbf{A}^1 , and let $e_{\tilde{L}}(\mathcal{G}_S)$ be the multiplicity of this ideal sheaf with respect to \tilde{L} . Then n.l.c. $w_S(m) = e_{\tilde{L}}(\mathcal{G}_S)$ by Corollary 3.3 of [4]. If $S = \{(S_i, \sigma_i)\}$ and $T = \{(T_j, \tau_j)\}$ are weighted bases of $H^0(C, L)$ and $H^0(C, M)$ respectively with L and M both of degree at least $2g + 1$, then we define $w_{(S,T)}(m)$ to be the least weight of a basis of $H^0(C, (L \otimes M)^{\otimes m})$ consisting of monomials in the tensors $S_i \otimes T_j$ (with weight $\sigma_i + \tau_j$). Such a basis exists by (iii) of Lemma 3.2. If S and T both have integer weights decreasing to zero, then Proposition 3.9 of [4] and Lemma 3.10 give respectively

$$\begin{aligned} \text{n.l.c. } (w_{(S,T)}(m)) &= e_{\tilde{L}}(\mathcal{G}_S) + 2e([\tilde{L}, \mathcal{G}_S], [\tilde{M}, \mathcal{G}_T]) + e_{\tilde{M}}(\mathcal{G}_T), \\ e([\tilde{L}, \mathcal{G}_S], [\tilde{M}, \mathcal{G}_T]) &\leq \frac{1}{2}(e_{\tilde{L}}(\mathcal{G}_S) + e_{\tilde{M}}(\mathcal{G}_T)). \end{aligned}$$

Hence we obtain

Proposition 3.3. *Suppose $S = \{(S_i, \sigma_i)\}$ and $T = \{(T_j, \tau_j)\}$ are weighted bases of $H^0(C, L)$ and $H^0(C, M)$ respectively such that the weights σ_i and τ_j both decrease to zero and such that L and M both have degree at least $2g + 1$. Then n.l.c. $(w_{(S,T)}(m)) \leq 2 \text{ n.l.c. } (w_S(m) + w_T(m))$.*

Note that by the homogeneity of this inequality we can allow the σ_i and τ_j to be rational. We will combine Proposition 3.3 and Lemma 3.2 to obtain an upper bound for $w_B(m)$ for each nice linesubbundle L of E . Fix a normalized weighted basis $B = \{(X_i, \sigma_i)\}$ of $H^0(C, E)$ and a nice subbundle L of E .

Recall that the associated long exact sequence is

$$0 \rightarrow H^0(C, L) \rightarrow H^0(C, E) \rightarrow H^0(C, M) \rightarrow 0.$$

Choose a basis $Y = \{Y_1, \dots, Y_n\}$ of $H^0(C, E)$ so that

$$(3.4) \quad \begin{aligned} (i) \quad & \text{span}\{Y_1, \dots, Y_n\} = V_i = \text{span}\{X_1, \dots, X_n\}, \\ (ii) \quad & Y = S \cup \tilde{T} \quad \text{where } S \text{ is a basis of } H^0(C, L). \end{aligned}$$

Let $B' = \{(Y_i, r_i)\}$. By Lemma 2.1, $w_B(m) = w_{B'}(m)$ so that in estimating $w_B(m)$ we may assume that B satisfies condition (3.4)(ii). We do so henceforth without comment and say the basis B is adapted to L . By Lemma 3.2(i) the image T of \tilde{T} in $H^0(C, M)$ forms a basis there. Let $S = \{S_1, \dots, S_{n_L}\}$, $\tilde{T} = \{\tilde{T}_1, \dots, \tilde{T}_{n_M}\}$ and $T = \{T_1, \dots, T_{n_M}\}$ ordered in each case so that the weights of the corresponding elements of B decrease.

Consider the diagram

$$\begin{array}{ccc} H^0(C, L) \otimes H^0(C, M) & \xrightarrow{\phi_{L,M}} & H^0(C, L \otimes M) \\ \psi \downarrow & & \parallel \\ \wedge^2 H^0(C, E) & \xrightarrow{\phi_E} & H^0(C, \wedge^2 E) \end{array}$$

where ψ is defined by $\psi(S_i \otimes T_j) = S_i \wedge \tilde{T}_j$. The diagram commutes, and the rows are surjective by (iii) and (iv) of Lemma 3.2. Define weights $\{s_i\}$ on S and $\{t_j\}$ on \tilde{T} and T so that the weight of each basis element equals the weight of the corresponding element of B . Then defining the weight of $R_{ij} = S_i \otimes T_j$ to be $s_i + t_j$ makes ψ weight preserving. We obtain a commutative diagram

$$\begin{array}{ccc} S^m(H^0(C, L) \otimes H^0(C, M)) & \longrightarrow & H^0(C, (L \otimes M)^{\otimes m}) \\ S^m \psi \downarrow & & \parallel \\ S^m \wedge^2 H^0(C, E) & \longrightarrow & H^0(C, (\wedge^2 E)^{\otimes m}) \end{array}$$

with surjective rows and with $S^m \psi$ weight preserving. Thus $w_B(m)$ is at most the minimum weight of a basis of $H^0(C, (L \otimes M)^m)$ consisting of monomials of degree m in the R_{ij} . Let $w_L = s_{n_L}$ and $w_M = t_{n_M}$, and define new weights σ_i and τ_j by $\sigma_i = s_i - w_L$ and $\tau_j = t_j - w_M$. Observe that one of w_L and w_M equals r_n which is zero since B is normalized, and that both the σ_i 's and the τ_j 's decrease to zero by the choice of the orderings on S and T . Let $S = \{(S_i, \sigma_i)\}$ and $T = \{(T_j, \tau_j)\}$ denote these weighted bases. As the (σ, τ) -weight of any of the R_{ij} differs from its (s, t) weight by $w_L + w_M$, the (σ, τ) -weight of a basis of $H^0(C, (L \otimes M)^m)$ consisting of monomials on the R_{ij} differs from its (s, t)

weight by $mh^0(C, (L \otimes M)^m)(w_L + w_M)$. Hence

$$w_B(m) \leq mh^0(C, (L \otimes M)^{\otimes m})(w_L + w_M) + w_{(S,T)}(m).$$

Applying Proposition 3.3 and taking leading coefficients gives

Theorem 3.5. *If L is a nice subbundle of E , and B is a normalized weighted basis of $H^0(C, E)$ adapted to L , then*

$$n.l.c. w_B(m) \leq 2d(w_L + w_M) + 2(n.l.c. w_S(m) + n.l.c. w_T(m)).$$

In the situation of the theorem, especially in §5, we will continue to use the notation developed in the preceding discussion (e.g., S, σ_i, w_L) to denote the quantities defined there.

4. Fix a weighted basis $B = \{(X_i, r_i)\}$ with associated weighted filtration $F_B = \{(V_i, r_i)\}$. We will give an estimate for n.l.c. $w_B(m)$ in terms of the subbundles of E generated by the sections in V_i . This criterion is an analogue for the rank-two case of estimates given for invertible sheaves in [2] and [6].

Let E_i be the subsheaf of E generated by the sections in V_i , $d_i = \deg E_i$, $e_i = d - d_i$, and let $s = s_B$ be the greatest index such that $\text{rank } E_i = 2$. If i and j are less than or equal to s , and $0 \leq k \leq m$, let $W_{i,j,k,N}$ be the image in $H^0(C, (\wedge^2 E)^{\otimes(m+1)N})$ of

$$S^N(S^{m-k}(\wedge^2 V_i) \vee S^k(\wedge^2 V_j) \vee \wedge^2 H^0(C, E)).$$

If $i \leq s$, let $W_{i,n,k,N}$ be the image of

$$S^N(S^{m-k}(\wedge^2 V_i) \vee S^k(V_i \wedge V_n) \vee \wedge^2 H^0(C, E)).$$

Lemma 4.1. *There is an N_0 depending only on the genus g of C such that if $N \geq N_0$ and $m \geq 0$, then:*

- (i) for $i, j \leq s$, $\dim W_{i,j,k,N} \geq N((m-k)d_i + kd_j)$,
- (ii) for $i \leq s$, $\dim W_{i,n,k,N} \geq N(m-k)d_i$.

Proof. We give the proof of (i), that for (ii) being similar. Since E_i is generated by the sections in V_i , $\wedge^2 E_i$ is generated by the sections in $\wedge^2 V_i$. Hence the elements of $W_{i,j,k,l}$ generate $L_{i,j,k} = (\wedge^2 E_i)^{m-k} \otimes (\wedge^2 E_j)^k \otimes \wedge^2 E$. Since $\wedge^2 E$ is very ample on C , and $\wedge^2 H^0(C, E)$ maps onto a very ample sublinear system of $\wedge^2 E$, $W_{i,j,k,l}$ forms a very ample sublinear system of $L_{i,j,k}$ without base points. Thus for N large, the elements of $W_{i,j,k,N}$ generate $H^0(C, L_{i,j,k}^{\otimes N})$ which by Riemann-Roch has dimension $N((m-k)d_i + kd_j + d) - g + 1$ from which the desired inequality is immediate. We omit the check that N can be chosen independent of C and E , which follows by arguments like those of Lemma 2.1 of [2].

Suppose a vector space V with a weighted filtration contains subspaces U_i satisfying:

- (i) $V = U_i \supset U_{i-1} \supset \cdots \supset U_1$,

(ii) $\text{codim } U_i = c_i$,

(iii) the weight of every element of U_i is at most w_i ,

(iv) $w_l \geq w_{l-1} \geq \dots \geq w_1$.

Then V has a basis of weight at most $\sum_{i=1}^{l-1} (w_{i+1} - w_i) c_i + w_1 \dim V$. Now pick a sequence of integers $1 = i_1 < i_2 < \dots < i_{l-1} < i_l = n$, where $i_{l-1} \leq s$, and apply this remark to the filtration of $H^0(C, (\wedge^2 E)^{\otimes (m+1)N})$ by $W_{i_1, i_2, 0, N} \supset W_{i_1, i_2, 1, N} \supset \dots \supset W_{i_1, i_2, m, N} = W_{i_2, i_3, 0, N} \supset W_{i_2, i_3, 1, N} \supset \dots \supset W_{i_{l-1}, i_l, m, N}$. The weight of any section in $W_{i, j, k, N}$ is bounded by $2N((m-k)r_i + kr_j + r_0)$ if $j \leq s$, and by $N(2(m-k)r_i + k(r_i + r_n) + 2r_0)$ if $j = n$. From Lemma 4.1, for $j \leq s$ we have

$$\begin{aligned} \text{codim } W_{i, j, k, N} &\leq (N(m+1)d - g + 1) - N((m-k)d_i + kd_j) \\ &\leq N(d + (m-k)e_i + ke_j), \end{aligned}$$

$$\begin{aligned} \text{codim } W_{i, n, k, N} &\leq (N(m+1)d - g + 1) - N(m-k)d_i \\ &\leq N((m-k)e_i + (k+1)d). \end{aligned}$$

Hence we obtain

$$\begin{aligned} w_B((m+1)N) &\leq \sum_{j=1}^{l-2} \sum_{k=0}^m 2N(r_{i_j} - r_{i_{j+1}}) (N((m-k)e_{i_j} + ke_{i_{j+1}} + d)) \\ &\quad + \sum_{k=0}^m N(r_{i_{l-1}} - r_n) (N((m-k)e_{i_{l-1}} + (k+1)d)) \\ &\quad + N(m(r_{i_{l-1}} + r_n) + 2r_0) ((m+1)Nd - g + 1) \\ &= \frac{(mN)^2}{2} \left[2 \sum_{j=1}^{l-2} (r_{i_j} - r_{i_{j+1}}) (e_{i_j} + e_{i_{j+1}}) \right. \\ &\quad \left. + (r_{i_{l-1}} - r_n) (e_{i_{l-1}} + d) + 2(r_{i_{l-1}} + r_n)d \right] + O(1), \end{aligned}$$

where in the $O(1)$ term we have collected all terms of order 1 in m . If we take B to be normalized so that $r_n = 0$, then by applying this to *all* subsequences of $(1, \dots, n)$ simultaneously and taking leading coefficients we obtain

Theorem 4.2. *If B is a normalized weighted basis of $H^0(C, E)$, then*

$$\begin{aligned} n.l.c. w_B(m) &\leq \min_{(1=i_1 < \dots < i_{l-1} \leq s)} \sum_{j=0}^{l-2} 2(r_{i_j} - r_{i_{j+1}}) (e_{i_j} + e_{i_{j+1}}) \\ &\quad + r_{i_{l-1}} (e_{i_{l-1}} + 3d). \end{aligned}$$

5. In this section we fix a smooth curve C and a rank-two bundle E of degree d on C . Our aim is to establish Proposition 2.5 and thereby to prove

Theorem 5.1. *There is an M depending only on g so that if $d \geq M$ and E is stable (resp. semistable), then (C, E) is Hilbert stable (resp. semistable).*

Proof. We assume E is semistable. Let $\alpha = g - 1$, and let $k = 10^6 \alpha^2$. We say a line bundle is good if $\deg L \geq k$. We divide the proof into two cases. In our first case, we assume

$$(5.1.1) \quad \text{rk } E_i = 2 \quad \text{for } i < n - k.$$

We first estimate $h^1(E_i)$ for $i \leq n - k$. E_i has rank two and at least $2g + 2$ sections. Let L_1 be the sublinebundle of E so that $S_1 \in H^0(L)$, and let $L_2 = E/L_1$. Then both L_1 and L_2 have sections, and at least one has $g + 1$ sections. Hence $h^1(L_i) \leq \alpha + 1 = g$, and $h^1(L_1)$ or $h^1(L_2)$ is zero. Since $h^1(E_i) \leq h^1(L_1) + h^1(L_2)$, we see

$$(5.1.2) \quad h^1(E_i) \leq \alpha + 1 \quad \text{if } i \leq n - k.$$

Next we claim

$$(5.1.3) \quad h^1(E_i) = 0 \quad \text{if } i < \frac{1}{2}n - 3\alpha.$$

Indeed, if $h^1(E_i) \neq 0$, then $E_i^{-1} \otimes \Omega^1$ has a section, and so E_i has a quotient of degree at most $2g - 2$. Thus E_i and hence E would have a subbundle of degree $d_i - 2\alpha$. Since E is semistable,

$$(5.1.4) \quad d \geq 2(d_i - 2\alpha).$$

But

$$(5.1.5) \quad \begin{aligned} d_i &= h^0(E_i) + 2\alpha - h^1(E_i) \\ &\geq (n - i + 1) + 2\alpha - \alpha - 1 \geq n - i + 1. \end{aligned}$$

Since $i < \frac{1}{2}n - 3\alpha$, we have

$$d_i \geq \frac{n}{2} + 3g - 2,$$

and by (5.1.4),

$$d \geq 2\left(\frac{n}{2} + g\right) = n + 2g,$$

which contradicts the fact that $d = n + 2\alpha$. Thus (5.1.3) is established.

We see from (5.1.5) that

$$e_i = d - d_i = d - (h^0(E_i) + 2\alpha - h^1(E_i)) \leq i - 1 + h^1(E_i),$$

since $n + 2\alpha = d$ and $h^0(E_i) \geq n - i + 1$.

Define ε_i and f_i by

$$\varepsilon_i = \begin{cases} \frac{2\alpha}{d}(i-1) & \text{if } i \leq \frac{n}{2} - 3\alpha, \\ \frac{2\alpha}{d}(i-1) - \frac{n}{d}(\alpha+1) & \text{if } \frac{n}{2} - 3\alpha < i \leq n-k, \end{cases}$$

$$(5.1.6) \quad f_i = \frac{d}{n}(i-1 - \varepsilon_i).$$

We have

$$(5.1.7) \quad \left(\frac{d}{n}(i-1 - \frac{2\alpha}{d}(i-1))\right) = i-1,$$

so

$$(5.1.8) \quad f_i \geq (i-1) + h^1(E_i) \geq e_i,$$

by (5.1.2) and (5.1.3).

Define

$$P_B(I) = 2 \min_{(1=i_1 < \dots < i_{l-1}=I)} \sum_{j=1}^{l-2} (r_{i_j} - r_{i_{j+1}})(e_{i_j} + e_{i_{j+1}}),$$

$$P(I) = 2 \min_{(1=i_1 < \dots < i_{l-1}=I)} \sum_{j=1}^{l-2} (r_{i_j} - r_{i_{j+1}})(f_{i_j} + f_{i_{j+1}}).$$

Then $P(I) \geq P_B(I)$. Further define

$$Q(I) = \max_{2 < i \leq I} \frac{f_i^2}{(i-1)f_i - \sum_{j=1}^{i-1} f_j}.$$

By Corollary 4.3 of [4],

$$P(I) \leq 2Q(I) \sum_{j=1}^I (r_j - r_I).$$

Thus

$$P_B(I) \leq 2Q(I) \sum_{j=1}^I (r_j - r_I).$$

Our next object is to estimate $Q(I)$. To this end, we define δ_i by

$$\delta_i = \frac{2d}{n} - \frac{f_i^2}{(i-1)f_i - \sum_{j < i} f_j} \quad \text{for } i \geq 2.$$

We wish to show $\delta_i \geq 1/2n$. If $i \leq n/2 - 3\alpha$, then $f_i = (i-1)$ and a direct computation shows that $\delta_i \geq 1/(2n)$. Assume $i > n/2 - 3\alpha$. First notice that we have

$$|f_i - i + 1| \leq \alpha + 1$$

from (5.1.6) and (5.1.7). Hence

$$\left| (i-1)f_i - \sum_{j<i} f_j - (i-1)^2 - \frac{1}{2}(i-1)(i-2) \right| \leq 2(\alpha+1)i.$$

So

$$(i-1)f_i - \sum f_j \leq \frac{1}{2}(i-1)i + 2(\alpha+1)i \leq \frac{1}{2}i(i+4\alpha+3).$$

We compute

$$\begin{aligned} \left(\frac{n}{d}\right)^2 ((i-1)f_i - \sum f_j) \delta_i &= 2((i-1)f_i - \sum f_j) - \frac{n}{d}(f_i^2) \frac{n}{d} \\ &= (i-1) + 2\sum \varepsilon_j - \varepsilon_i^2. \end{aligned}$$

We next claim that

$$(5.1.10) \quad 2\sum_{j<i} \varepsilon_j - \varepsilon_i^2 > -18\alpha^2,$$

for $i > n/2 - 3\alpha$. Once (5.1.10) is established, we will have

$$\delta_i \geq \frac{(i-1-18\alpha^2)d^2}{((i-1)f_i - \sum f_j)n^2} \geq \frac{2(i-18\alpha^2-1)d^2}{i(i+4\alpha+3)n^2} \geq \frac{1}{2n}.$$

Thus

$$(5.1.11) \quad \delta_i \geq \frac{1}{2n}.$$

Since (5.1.11) holds for $i \leq n/2 - 3\alpha$, (5.1.11) holds in general.

We next establish our claim (5.1.10). Let J be the greatest integer in $n/2 - 3\alpha$. Then

$$\begin{aligned} d \sum_{j=1}^{i-1} \varepsilon_j &= 2\alpha \sum_{j=1}^{i-1} (j-1) - n(\alpha+1)(i-J-1) \\ &\geq \alpha((i-1)(i-2) - 2n(i-J-1)). \end{aligned}$$

The function $f(i) = (i-1)(i-2) - 2n(i-J-1)$ has its minimum when $2i-3=2n$. Thus since $i \leq N-k$ and $k > 10^6\alpha^2$,

$$\begin{aligned} f(i) &\geq (n-k-1)(n-k-2) - 2n(n-k-\frac{n}{2}+3\alpha-1) \\ &= -(6\alpha+1)n \geq -7\alpha^2 n. \end{aligned}$$

Also, for $n/2 - 3\alpha < i \leq n-k$, $-2 \leq \varepsilon_i \leq 2\alpha$. So

$$2\sum_{j=1}^{i-1} \varepsilon_j - \varepsilon_i^2 \geq -18\alpha^2.$$

Thus if (C, E) is not stable with respect to B , we would have for each I

$$\frac{4d}{n} \leq Q(I) \left(\sum_{j=1}^I (r_j - r_I) \right) + r_I(e_I + 3d).$$

From (5.1.11), we see

$$\frac{f_i^2}{(i-1)f_i - \sum f_j^2} = \frac{2d}{n} - \delta_i \geq \left(\frac{2d}{n} - \frac{1}{2n} \right).$$

So

$$Q(I) \leq \frac{2d}{n} - \frac{1}{2n}.$$

Thus

$$(5.1.12) \quad \frac{4d}{n} \leq \left(\frac{4d-1}{n} \right) \left(\sum_{j \leq I} r_j - r_I \right) + r_I(f_I + 3d).$$

Next let $\beta(I) = 1 - \sum_{i=1}^I r_i$. Since $\sum r_i = 1$, we can write (5.1.12) as

$$r_I \left(f_I + 3d - \frac{4d}{n} I \right) \geq \frac{4d}{n} \beta(I) + \frac{1}{n} \sum_{j \leq I} (r_j - r_I).$$

Now

$$f_I = \frac{d}{n} ((I-1) - \varepsilon_I), \quad -\varepsilon_I \leq \frac{1}{d} (n + 6g^2) \leq 2.$$

So

$$f_I + 3d - \frac{4d}{n} I \leq \frac{d}{n} (3(n-I) + 1).$$

Thus

$$(5.1.13) \quad r_I (3(n-I) + 1) \geq 4\beta(I) + \frac{1}{d} \sum_{j \leq I} (r_j - r_I).$$

In particular,

$$(5.1.14) \quad r_I (3(n-I) + 1) \geq 4\beta(I).$$

Let $J_I = n - 10^l k$ where $k = 10^6 \alpha^2$.

We claim

$$(5.1.15) \quad (k+2)r_{J_0} \leq \frac{1}{100n}.$$

Indeed, note for any J ,

$$\beta(n - 10J) \geq 9Jr_{n-J} + \beta(n - J).$$

From (5.1.14),

$$r_{n-10J} \geq \frac{4}{3} \frac{\beta(n-10J)}{10J+1} \geq \frac{4}{3} \frac{9J}{10J+1} r_{n-J} \geq \frac{12}{11} r_{n-J}.$$

So $r_{J_i} \geq (12/11)^l r_{J_0}$. Choose l so that $(12/11)^l \geq 300(k+2)$ and $J_l \geq 2n/3$. (Recall that we are assuming that d and n are large with respect to g and hence to k .)

$$1 \geq \sum_{j=1}^{\lfloor 2n/3 \rfloor} r_i \geq \frac{n}{2} r_{J_l} \geq \frac{n}{2} (300(k+2)) r_{J_0}.$$

Thus our claim (5.1.15) is established.

Next note that

$$\sum (r_i - r_j) = 1 - \beta(I) - I r_I,$$

so (5.1.13) shows that

$$r_I \left(3(n-I) + 1 + \frac{I}{d} \right) \geq 4\beta(I) + \frac{1}{d} (1 - \beta(I)) \geq \frac{1}{d}.$$

Finally, we take $I = J_0$. Then

$$\frac{3k+2}{100n(k+2)} \geq \frac{1}{d},$$

which contradicts $d = n + 2\alpha$. Thus we have established Theorem 5.1 under assumption (5.1.1).

We may accordingly assume $\text{rk } E_{n-k} = 1$ and hence $\text{rk } E_i = 1$ for $i \geq n-k$.

Let L be the sublinebundle of E containing E_i for $i \geq n-k$. We may replace B by a basis adapted to L without affecting the hypothesis. If l is the greatest integer so that $S_l \in H^0(L)$, then $l \geq n/2$ since otherwise L would have more than $n/2$ sections, thus contradicting the semistability of E . Thus $w_M \geq 2/n$ with strict inequality if E is stable.

Recall from Theorem 3.5 that $\text{n.l.c. } w_B(m) \leq 2(w_L)d + 2\text{n.l.c.}(w_S(m) + w_T(m))$. Since L is good, d_L and d_M are greater than K , and it follows from Corollary 4.6 of [4] that $\text{n.l.c. } w_S(m) \leq 2 \sum_{i=1}^{n_L} \sigma_i$ and $\text{n.l.c. } w_T(m) \leq 2 \sum_{j=1}^{n_M} \tau_j$. Note that

$$1 = \sum_{i=1}^n r_i = n_M w_M + \sum_{i=1}^{n_L} \sigma_i + \sum_{j=1}^{n_M} \tau_j.$$

If E is stable we obtain

$$\begin{aligned} \text{n.l.c. } w_B(m) &\leq 2w_M d + 4 \left(\sum_{i=1}^{n_L} \sigma_i + \sum_{i=1}^{n_M} \tau_j \right) \\ &= 2w_M d + 4(1 - n_M w_M) < \frac{4d}{n} - 2w_M(2n_M - n). \end{aligned}$$

If E is semistable, then $n_M \geq n/2$, hence $w_M \leq 2/n$. Unless $n_M w_M = 1$, this implies

$$\text{n.l.c. } w_B(m) < \frac{4d}{n} - 2dw_M(2n_M - n) \leq \frac{4d}{n},$$

so that (C, E) is stable with respect to B . If $n_M w_M = 1$, this argument only shows that $\text{n.l.c. } w_B(m) \leq 4d/n$ which does not suffice to prove (C, E) semistable with respect to B . However, in this case all the σ_i 's and τ_j 's must be zero. Hence every section $R_{i_j} = S_i \otimes T_j$ has weight w_M . But then

$$w_B(m) \leq mh^0(C, L \otimes M)^{\otimes m} w_M \leq \frac{2mh^0(C, (\wedge^2 E)^{\otimes m})}{h^0(C, E)},$$

since $w_M \leq 1/n_M \leq 2/n$. This completes the proof of Proposition 2.5.

Now Theorem 5.1 follows from Corollary 2.3. In fact, if E is unstable, L is the destabilizing line subbundle, and B is any standard basis whose filtration is $W \supset H^0(C, L) \supset \{0\}$, then φ_3 kills all elements of nonpositive weight, hence so does each φ_C^m . Therefore $w_B(m) > 0$, and (C, E) is Hilbert unstable. Hence Theorem 1.1 is proved.

6. We continue to suppose that $d \geq 1000g(g - 1)$. Our object is to prove

Proposition 6.1. *There is an M (depending on d) so that if $m \geq M$, and φ_C^m is semistable for $C \in S_{g,d}$, then C is semistable as a curve.*

We begin with a few general definitions. Let \mathcal{F} be a coherent sheaf on a scheme, and let $W \subseteq H^0(X, \mathcal{F})$ be a subspace so that \mathcal{F} is generated at each point by sections in W .

Definition 6.2. A weighted filtration on \mathcal{F}

$$B = \begin{pmatrix} \mathcal{F}_k & \mathcal{F}_{k-1} & \cdots & \mathcal{F}_1 \\ r_k & r_{k-1} & \cdots & r_1 \end{pmatrix}$$

is a sequence of subsheaves

$$\mathcal{F}_k \subseteq \mathcal{F}_{k-1} \subseteq \cdots \subseteq \mathcal{F}_1 = \mathcal{F}$$

and rational numbers $r_i, r_k \leq r_{k-1} \leq \cdots \leq r_1$. (Note: In the rest of this paper, filtrations will increase from left to right.)

If

$$B' = \begin{pmatrix} \mathcal{F}'_i \\ r'_i \end{pmatrix}$$

is another weighted filtration on \mathcal{F} , and if it happens that $\mathcal{F}'_i \subseteq \mathcal{F}_i$ whenever $r_i \leq r'_i$, we say B' dominates B .

Let $\pi: Y \rightarrow X$ be a map. Given a weighted filtration $B = \begin{pmatrix} \mathcal{G}_i \\ r_i \end{pmatrix}$ on $\pi^*(\mathcal{F})$, there is an induced filtration $B' = \begin{pmatrix} W_i \\ r'_i \end{pmatrix}$ on W , where

$$W_i = \{s \in W \mid \pi^*(s) \in H^0(Y, \mathcal{G}_i)\}.$$

Conversely, given a weighted filtration on W , there is an induced filtration on $\pi^*(\mathcal{F})$, where \mathcal{G}_i is the subsheaf of $\pi^*(\mathcal{F})$ generated by W_i .

The weight of a filtration $\begin{pmatrix} W_i \\ r_i \end{pmatrix} = B$ on W is $\sum \dim(W_i/W_{i-1})r_i = w(B)$.

Now let $\varphi: \mathcal{F} \rightarrow \mathcal{G}$ be a map of coherent sheaves. The weighted filtration

$$\begin{pmatrix} \ker \varphi & \mathcal{F} \\ 0 & 1 \end{pmatrix}$$

will be denoted

$$(6.2.1) \quad [\mathcal{F} \rightarrow \mathcal{G}].$$

Now let L be a line bundle on a curve C , and let $V \subseteq H^0(C, L)$ be a very ample linear system. Let $\begin{pmatrix} V_i \\ r_i \end{pmatrix} = B$ be a weighted filtration on V . Choose a compatible weighted basis $\{(X_j, \rho_j)\}$ of V , and let $w_B(m, C)$ be the minimum weight of a basis of $H^0(C, L^{\otimes m})$. Then $w_B(m, C)$ is a polynomial in m for $m \gg 0$.

Now suppose that C is a curve on G and that $\begin{pmatrix} W_i \\ r_i \end{pmatrix}$ is a weighted filtration on W . There is an induced weighted filtration B' on the image V of $\wedge^2 W$ in $H^0(C, \det \mathcal{E}_C)$. If V is very ample, we define $w_B(m, C) = w_{B'}(m, C)$.

For the remainder of this section, we consider a curve C , a very ample linear system $V \subseteq H^0(C, L)$ and a weighted filtration $B = \begin{pmatrix} V_i \\ r_i \end{pmatrix}$. Our aim is to give two useful estimates for n.l.c. $w_B(m, C)$.

Lemma 6.4. *Suppose $C_i \subseteq C$ are subcurves of C , and the natural map $\varphi: \mathcal{O}_C \rightarrow \bigoplus \mathcal{O}_{C_i}$ has kernel and cokernel of finite length. Then*

$$\text{n.l.c. } w_B(m, C) \geq \sum_i \text{n.l.c. } w_B(m, C_i).$$

Proof. Let q be the maximum of the lengths of the kernel and cokernel of φ . Then for $m \gg 0$, the kernel and cokernel of

$$\varphi_m: H^0(C, L^{\otimes m}) \rightarrow \bigoplus H^0(C_i, L^{\otimes m})$$

have dimension $\leq q$. Given a basis P_1, \dots, P_t of $H^0(C, L^{\otimes m})$, we can suitably reorder the P_i and partition P_1, \dots, P_{t-q} into sets $Q_i \subseteq \{P_1, \dots, P_{t-q}\}$ so that Q_i gives an independent set in $H^0(C_i, L^{\otimes m})$. Thus

$$w_B(m, C) - mr_1q \geq \sum w_B(m, C_i) - mr_kq.$$

Taking normalized leading coefficients yields the lemma.

Now suppose C is irreducible. Let $\pi: \tilde{C} \rightarrow C$ be the normalization of C_{red} ; and let $\mathcal{I} \subseteq \mathcal{O}_C$ be the ideal of C_{red} . Let l be the length of the local ring of the generic point of C . Suppose R is an effective divisor on \tilde{C} . Let $B = (V_i)$ be a weighted filtration and let p be an integer and suppose the r_i are integers.

Proposition 6.5. *Suppose that V_j maps to zero in $H^0(\tilde{C}, \tilde{L})$ for $j > p$ and that V_1 maps to $H^0(\tilde{C}, \tilde{L}((-r_1 + r_i)R))$. If $\deg L \geq (r_1 - r_p)\deg R$, then we have*

$$\text{n.l.c. } w_B(m, C) \geq (r_1 - r_p)^2 \deg R + 2lr_p \deg \tilde{L}.$$

Proof. First, replace C by the subscheme defined by \mathcal{I}^l . Since \mathcal{I}^l is supported at a finite number of points, neither the hypothesis nor conclusion of the theorem are changed.

Let B' be the weighted filtration

$$\begin{pmatrix} V_p & \cdots & V_1 \\ r_p & \cdots & r_1 \end{pmatrix},$$

that is, we change the weights of the V_i for $i \geq p$ from r_i to r_p . Now let $\{(X_i, \rho_i)\}$ be a basis of V compatible with B . Let M be a monomial in the X_i 's which is nonzero in $H^0(C, L^{\otimes m})$. Then M can involve at most l of X_i 's with $X_i \in V_p$, since $\mathcal{I}^l = 0$. Thus

$$\text{n.l.c. } w_B(m, C) = \text{n.l.c. } w_{B'}(m, C),$$

since the B and B' weights of a monomial differ by at most $l(r_p - r_k)$, where r_k is the lowest weight in B . Hence we may assume $B = B'$.

Next, notice that

$$h^0(C, L^{\otimes m}) = ml \deg \tilde{C} \tilde{L} + O(1),$$

since $\mathcal{I}^{k-1}/\mathcal{I}^k$ is nonzero at the generic point of C_{red} for $k = 1, \dots, l$. Consider a new weighted filtration

$$B' = \begin{pmatrix} V_i \\ r_i - r_p \end{pmatrix}.$$

Then

$$\begin{aligned} w_B(m, C) &= w_{B'}(m, C) + mr_p h^0(C, L^{\otimes m}) \\ &= w_{B'}(m, C) + m^2 r_p l \deg \tilde{L} + O(m). \end{aligned}$$

Hence it suffices to prove Proposition 6.5 for $r_p = 0$.

Since $r_i \geq 0$,

$$w_B(m, C) \geq w_B(m, C_{\text{red}}),$$

so we may assume C is reduced. Now let M be any monomial in $V^{\otimes m}$ of weight Q . Then the image of M is in $H^0(\tilde{C}, \tilde{L}^{\otimes m}((Q - r_1 m)R))$. Thus there is a constant C_1 so that the image of an M of weight Q lies in a subspace of codimension at least $(r_1 m - Q)\deg R - C_1$ in $H^0(C, L^{\otimes m})$. Adding up the possible contributions for each weight Q , we see any basis must have weight at least

$$\sum_{Q=0}^{mr_1} [Q \deg R + O(1)] = r_1^2 \deg R \frac{m^2}{2} + O(m).$$

7. Let $C \in S_{g,d}$. We can find curves $C_i \subseteq C$ and integers l_i so that the following hold:

(7.1.1) Each C_i is irreducible.

(7.1.2) $\mathcal{G}_{C_i}^{l_i} = 0$, where \mathcal{G}_{C_i} is the ideal of C_i in C .

(7.1.3) l_i is the length of the local ring of the generic point of C_i .

(7.1.4) The natural map $\mathcal{O}_C \rightarrow \bigoplus \mathcal{O}_{C_i}$ has kernel and cokernel of finite length.

Given a weighted filtration B on W , Lemma 6.4 shows that

$$\text{n.l.c. } w_B(m, C) \geq \sum \text{n.l.c. } w_B(m, C_i).$$

Now let $E = \mathcal{O} \otimes \mathcal{O}_C$, let \tilde{C}_i be the normalization of $(C_i)_{\text{red}}$, and let $\pi_i: \tilde{C}_i \rightarrow C$ be the induced map. Let $\tilde{E}_i = \pi_i^*(E)$ and let $d_i = \deg_{\tilde{C}_i} \tilde{E}_i$. Let B be a weighted filtration on W . If B_i is a weighted filtration on \tilde{E}_i , we say B dominates B_i if the filtration induced from B on \tilde{E}_i dominates B_i .

Lemma 7.2. *Let R be an effective divisor on \tilde{C}_i , and let $k = \deg_{C_i} E - 2 \deg R$. Suppose B dominates*

$$\begin{pmatrix} \tilde{E}_i(-R) & E_i \\ 0 & 1 \end{pmatrix}.$$

If $k \geq 0$, then

$$(7.2.1) \quad \text{n.l.c. } w_B(m, C_i) \geq 4 \deg R,$$

while if $k + \deg R \geq 0$ and $k < 0$, then

$$(7.2.2) \quad \text{n.l.c. } w_B(m, C_i) \geq \deg R + 2l_i d_i.$$

Proof. If $k \geq 0$, the filtration induced by W on $\wedge^2 E$ dominates

$$\begin{pmatrix} \wedge^2 \tilde{E}_i(-2R) & \wedge^2 \tilde{E}_i(-R) & \wedge^2 \tilde{E}_i \\ 0 & 1 & 2 \end{pmatrix}.$$

Applying Proposition 6.5 gives (7.2.1).

If $k + \deg R \geq 0$ and $k < 0$, the filtration induced by W on $\wedge^2 E$ dominates

$$\begin{pmatrix} \wedge^2 \tilde{E}_i(-R) & \wedge^2 \tilde{E}_i \\ 1 & 2 \end{pmatrix},$$

since $H^0(C, \wedge^2 \tilde{E}_i(-2R)) = 0$. Applying Proposition 6.5 gives

$$\text{n.l.c. } w_B(m, C_i) \geq \deg R + 2l_i d_i.$$

Lemma 7.3. *Let E' be a rank-two subsheaf of \tilde{E}_i with $\deg E' \geq 0$. Suppose B dominates*

$$\begin{pmatrix} E' & \tilde{E}_i \\ 0 & 1 \end{pmatrix}.$$

Then

$$\text{n.l.c. } w_B(m, C_i) \geq d_i - \deg E'.$$

Proof. The filtration induced on $\wedge^2 \tilde{E}_i$ dominates

$$\begin{pmatrix} \wedge^2 E' & \wedge^2 \tilde{E}_i \\ 0 & 1 \end{pmatrix}.$$

Now $\wedge^2 E' = \wedge^2 \tilde{E}_i(-R)$, where $\deg R = d_i - \deg E'$. Proposition 6.4 applies.

Lemma 7.4. *Suppose that $0 \rightarrow M \rightarrow \tilde{E}_i \rightarrow L \rightarrow 0$ is exact with M and L invertible and that B dominates*

$$\begin{pmatrix} M(-R) & \tilde{E}_i \\ 0 & 1 \end{pmatrix}.$$

Then

$$\text{n.l.c. } w_B(m, C_i) \geq \deg R + 2l_i d_i,$$

if $\deg R \leq \deg \tilde{E}_i$.

Proof. The induced filtration on $\wedge^2 \tilde{E}_i$ dominates

$$\begin{pmatrix} \wedge^2 \tilde{E}_i(-R) & \wedge^2 \tilde{E}_i \\ 1 & 2 \end{pmatrix}.$$

Lemma 7.5. *If B dominates*

$$\begin{pmatrix} 0 & \tilde{E}_i \\ 0 & 1 \end{pmatrix},$$

then n.l.c. $w_B(m, C_i) \geq 4l_i d_i$.

Proof. Left to reader.

Now write $d/n = 1 + \varepsilon$. Since $n = d + 2(1 - g)$ and $n \geq 1000g(g - 1)$, we see $\varepsilon \leq 1/998g$. Let B be a weighted filtration on W . We will say B is destabilizing if

$$\text{n.l.c. } w_B(m, C) > 4(1 + \varepsilon)w(B).$$

Throughout the rest of the section, we will assume $C \in S_{g,d}$ has no destabilizing flags. Our aim in this section is to establish that $l_i = 1$.

Lemma 7.6. *If \tilde{E}_i has a trivial quotient $\tilde{E}_i \rightarrow \mathcal{O} \rightarrow 0$, then $l_i = 1$ and $d_i = 1$.*

Proof. We consider the filtration B induced on W by $[\tilde{E}_i \rightarrow \mathcal{O}]$ in the notation of (6.2.1).

Lemma 7.4 with $R = \emptyset$ gives

$$(7.6.1) \quad \text{n.l.c. } w_B(m, C_i) \geq 2l_i d_i.$$

On the other hand, if there is a component C_j meeting C_i , Lemma 7.3 shows

$$\text{n.l.c. } w_B(m, C_j) \geq 1.$$

Hence from (7.6.1),

$$4(1 + \varepsilon) > \text{n.l.c. } w_B(m, C) \geq \text{n.l.c. } w_N(m, C_i) \geq 2l_i d_i.$$

Hence $l_i d_i \leq 2$, so C_i must meet some C_j . Thus

$$(1 + \varepsilon) \geq \frac{1}{2}l_i d_i + \frac{1}{4},$$

which shows $l_i d_i = 1$. The same method of proof shows

Corollary 7.6.2. *If $C' \subseteq C$ is a curve, and $E_{C'}$ has a trivial quotient, then C' has one component, and $E_{C'}$ has degree 1.*

Lemma 7.7. $l_i = 1$ for all i .

Proof. Suppose $l_i \geq 2$. Let B be the weighted filtration on W induced by

$$\begin{pmatrix} 0 & \tilde{E}_i \\ 0 & 1 \end{pmatrix}.$$

First, suppose B is the trivial filtration, i.e.,

$$B = \begin{pmatrix} 0 & W \\ 0 & 1 \end{pmatrix}.$$

Then the map from W to $H^0(\tilde{E}_i)$ is injective. Since $\sum l_j d_j = d$, we have $d_i \leq \frac{1}{2}d$. Hence

$$d + 2(1 - g) \leq h^0(\tilde{E}_i) \leq \deg \tilde{E}_i + 2 \leq \frac{d}{2} + 2,$$

which is impossible.

The total weight of B is less than or equal to $h^0(\tilde{E}_i) \leq d_i + 2$. Hence

$$(7.7.1) \quad (1 + \varepsilon)(d_i + 2) \geq (1 + \varepsilon)h^0(\tilde{E}_i) \geq l_i d_i + \frac{\delta}{4},$$

where $\delta = \sum_{j \neq i} w_B(m, C_j) \geq 0$. We reach a contradiction if $l_i \geq 3$ or $d_i \geq 3$. So we may assume $l_i = 2$ and $d_i \leq 2$.

Now $\deg_{C_i} \wedge^2 E \leq 4$, so C_i must meet another component C_j . Suppose $P \in \tilde{C}_j$ maps to $C_i \cap C_j$. Then the filtration on \tilde{E}_j induced by B dominates

$$\begin{pmatrix} \tilde{E}_j(-P) & \tilde{E}_j \\ 0 & 1 \end{pmatrix}.$$

Applying (7.2.1) if $d_j \geq 2$, and (7.2.2) if $d_j = 1$, we see

$$\text{n.l.c. } w_B(m, C_j) \geq \begin{cases} 4 & \text{if } d_j \geq 2, \\ 3 & \text{if } d_j = 1. \end{cases}$$

Now if $d_j = 1$, then either C_i or C_j must meet another component C_k , and Lemma 7.3 shows that

$$\text{n.l.c. } w_B(m, C_k) \geq 1.$$

In either case, $\delta \geq 4$. This contradicts (7.7.1) if $l_i \geq 2$ and $d_i = 2$. If $l_i \geq 2$ and $d_i = 1$, then C_i is P^1 , and hence E_{C_i} has a trivial quotient, contradicting Lemma 7.6. Thus $l_i = 1$ in all cases.

8. Our aim in this section is to show that C_{red} has only nodes as singularities.

Let $C' \subseteq C_{\text{red}}$ be a curve.

Lemma 8.1. *If $h^0(C', E) \leq \deg_{C'} E$, then $\deg_{C'}(E) \geq 20g$.*

Proof. Suppose not. Then some component C_j of C must meet C' as we are assuming $d \geq 1000g(g - 1)$. Consider the weighted filtration B given by $[E \rightarrow E_{C'}]$. Then

$$\begin{aligned} \text{n.l.c. } w_B(m, C) &\geq \text{n.l.c. } w_B(m, C') + \text{n.l.c. } w_B(m, C_j) \\ &\geq 4 \deg_{C'}(E) + 1, \end{aligned}$$

by (7.5) and (7.3) respectively. But

$$\begin{aligned} w(B) &= h^0(C', E) \geq \deg_{C'}(E), \\ \text{n.l.c. } w_B(m, C) &\leq 4(1 + \varepsilon)w(B). \end{aligned}$$

Combining these gives

$$4(1 + \varepsilon)\deg_{C'}(E) \geq 4\deg_{C'}(E) + 1,$$

which is impossible if $\deg_{C'}(E) < 20g$.

Lemma 8.2. *Let $C' \subseteq C_{\text{red}}$ be a curve and let C'' be a component of C' . Then there is a short exact sequence*

$$0 \rightarrow L \rightarrow E_{C'} \rightarrow M \rightarrow 0,$$

where L and M are invertible, L and M have nonnegative degree on each component of C' , and $\deg_{C''} L > 0$.

Proof. Let P_1, \dots, P_k be the singular points of C' and let $E' = E_{C'}$. Let Z_i be the common zeros of sections of E' which vanish at P_i . Then Z_i is a finite set, since if $Z_i \supseteq C_j$, the dimension of the image of $H^0(E')$ in $H^0(C_j, E')$ would be at most one. But $\wedge^2 E$ is very ample. By picking a point $P \in C''$ not in any Z_i , we can find a section s which vanishes at P , but not at any singular point. We then let L be the smallest subbundle of E containing S to establish our lemma.

Corollary 8.2.1. *Suppose every line bundle L in $E_{C'}$, which has positive total degree and nonnegative degree on each component of C' , satisfies $h^0(C', L) \leq \deg_{C'} L$. Then $\deg_{C'} E \geq 20g$.*

Proof. We write

$$0 \rightarrow L \rightarrow E_{C'} \rightarrow M \rightarrow 0.$$

Since $E_{C'}$ is generated by global sections, M has nonnegative degree on each component of C' . If $\deg_{C'}(M) = 0$, $E_{C'}$ has a trivial quotient, so Corollary 7.6.2 shows C' is smooth and rational, and the hypothesis of Corollary 8.2.1 fails. Hence

$$h^0(C', L) \leq \deg_{C'}(L),$$

$$h^0(C', M) \leq \deg_{C'}(M).$$

So

$$h^0(C', E) \leq \deg_{C'}(E),$$

and Lemma 8.1 applies.

Lemma 8.3. *Let P be a point of C_i . Then the map $\pi_i: \tilde{C}_i \rightarrow C$ is unramified at P .*

Proof. Suppose not. Let $Q = \pi_i(P)$. Then every section of $\mathcal{O}_{C,Q}$ vanishing at Q vanishes at least twice at P . Thus the hypothesis of Corollary 8.2.1 is satisfied since $(C_i)_{\text{red}}$ is singular. Hence $\deg_{C_i} E \geq 20$.

Now consider the filtration on W

$$B = \begin{pmatrix} W_3 & W_2 & W_1 \\ 0 & 1 & 3 \end{pmatrix}$$

induced by

$$\begin{pmatrix} \tilde{E}_i(-3P) & \tilde{E}_i(-2P) & \tilde{E}_i \\ 0 & 1 & 3 \end{pmatrix}.$$

Now $\dim W_1/W_2 \leq 2$ as the map from \tilde{C}_i to C is ramified at P . Further $\dim W_2/W_3 \leq 2$. Hence $w(B) \leq 8$. On the other hand, the induced filtration on $\wedge^2 \tilde{E}_i$ is

$$\begin{pmatrix} (\wedge^2 \tilde{E}_i)((-6+k)P) \\ k \end{pmatrix}.$$

Proposition 6.5 shows that n.l.c. $w_B(m, C) \geq 36$. So $4(1 + \epsilon)8 \geq 36$, a contradiction.

Lemma 8.4. C_{red} has no triple points.

Proof. Suppose three distinct components, say C_1, C_2, C_3 , meet at a point P . We let B be the weighted filtration on W induced by $[E \rightarrow E_P]$. Then $w(B) \leq 2$. Now (7.2.1) and (7.2.2) show that

$$\text{n.l.c. } w_B(m, C_i) \geq 3,$$

for $i = 1, 2, 3$ and n.l.c. $w_B(m, C_i) \geq 0$ for $i > 3$ and therefore

$$\text{n.l.c. } w_B(m, C) \geq 9$$

by (6.4). Hence $4(1 + \epsilon)2 > 9$, a contradiction.

Now if C_1 and C_2 meet at a singular point $P \in C_1$, then $\deg C_1 \geq 20$. Using (7.2.1) applied to C_1 and $R = \pi_1^{-1}(P)$, we see

$$\text{n.l.c. } w_B(m, C_1) \geq 8,$$

and we obtain a contradiction as before.

Similarly, C_1 cannot have a triple point.

Lemma 8.5. C has no tacnodes.

Proof. Suppose that C_1 and C_2 meet at P , and that the tangent lines of C_1 and C_2 are identical. Then the two weighted filtrations induced on W by

$$B_i = \begin{pmatrix} \tilde{E}_i(-2P) & \tilde{E}_i(-P) & \tilde{E}_i \\ 0 & 1 & 2 \end{pmatrix}$$

for $i = 1, 2$ are identical. Call this filtration B .

We may assume $d_1 \leq d_2$. Now if $d_1 = 1$, then C_1 is rational and $E_{C_1} \cong \mathcal{O} \oplus \mathcal{O}(1)$. Thus the map from $H^0(C_1, E(-P))$ to $E(-P) \otimes k_P$ is not surjective. So $w(B) \leq 5$ if $d_1 = 1$, and $w(B) \leq 6$ if $d_1 > 1$.

Now $C_1 \cup C_2$ satisfies the hypothesis of Lemma 8.1, so $d_1 + d_2 \geq 20g \geq 40$, and hence $d_2 \geq 4$. Applying Proposition 6.5, we see that

$$\text{n.l.c. } w_B(m, C_i) \geq 16,$$

if $d_i \geq 4$. On the other hand, if $d_i \leq 4$, the filtration induced by W on $\wedge^2 \tilde{E}_1$ dominates

$$\begin{pmatrix} \wedge^2 \tilde{E}_1(-d_1 P) & \cdots & \wedge^2 \tilde{E}_1(-P) & \wedge^2 \tilde{E}_1 \\ 4 - d_1 & \cdots & 3 & 4 \end{pmatrix},$$

since $H^0(C_1, \wedge^2 E((-d_1 - 1)P)) = 0$. Applying Proposition 6.5,

$$\text{n.l.c. } w_B(m, C_1) \geq d_1^2 + 2(4 - d_1)d_1 \geq d_1(8 - d_1).$$

Thus if $d_1 = 1$, then

$$4(5)(1 + \varepsilon) \geq \text{n.l.c. } w_B(m, C) \geq 16 + 7 = 23,$$

a contradiction. If $d_1 \geq 2$, then

$$4(6)(1 + \varepsilon) \geq \text{n.l.c. } w_B(m, C) \geq 16 + 12 = 28,$$

a contradiction. So C_1 and C_2 cross transversally.

Finally, if C_1 has a tacnode, then $d_1 \geq 8$. A similar argument produces a contradiction once again.

We have established

Proposition 8.6. C_{red} has only nodes as singularities.

9. Our main aim in this section is to establish that C is semistable as a curve, and that the map $W \rightarrow H^0(C, E)$ is an isomorphism.

We begin with a version of Clifford's Theorem following Saint-Donat.

Lemma 9.1. *Let D be a reduced curve with only nodes, and let L be a line bundle on D generated by global sections. If $H^1(D, L) \neq 0$, there is a curve $C' \subseteq D$ so that*

$$h^0(C', L) \leq \frac{1}{2} \deg_C L + 1.$$

Proof. Since $H^1(D, L) \neq 0$, $H^0(L^{-1} \otimes \omega_D) \neq 0$. So there is a nonzero $\varphi: L \rightarrow \omega_D$. We can find a curve $C' \subseteq D$ so that φ is not identically zero on each component of C' , but φ vanishes at all points $C' \cap \overline{D - C'} = \{P_1, \dots, P_k\}$. Since $\omega_{C'} = \omega_D(-P_1 \cdots -P_k)$, we actually obtain

$$\varphi: L_{C'} \rightarrow \omega_{C'}.$$

Choose a basis s_1, \dots, s_r of $\text{Hom}(L_{C'}, \omega_{C'})$ so that $\varphi = s_1$. We can choose a basis $t_1 \cdots t_p$ of $H^0(L_{C'})$ so that t_1 does not vanish at the zeros of s_1 nor at any singular point of C' . Suppose

$$a_1 \langle s_1, t_1 \rangle + a_2 \langle s_1, t_2 \rangle + \cdots = b_2 \langle s_2, t_1 \rangle + b_3 \langle s_3, t_1 \rangle + \cdots,$$

where the pairing $\langle s, t \rangle$ is into $H^0(C', \omega_{C'})$. Then $\langle s_1, t \rangle = \langle s, t_1 \rangle$, where $t \in H^0(C', L_{C'})$, and s is a linear combination of s_2, \dots, s_r . Since t vanishes where t_1 does, t is a multiple of t_1 . Hence s is a multiple of s_1 , contradicting the independence of the s_i 's. So

$$\begin{aligned} h^0(L_{C'}) + h^0(\omega_{C'} \otimes L_{C'}^{-1}) &\leq g + 1, \\ h^0(L_{C'}) - h^0(\omega_{C'} \otimes L_{C'}^{-1}) &\leq \text{deg}_{C'}(L) + 1 - g. \end{aligned}$$

Adding the above two inequalities thus gives the desired result.

Lemma 9.2. *Let C' be a proper subcurve of C_{red} . Then*

$$h^0(C', E) > \text{deg}_{C'}(E) + 2(1 - g).$$

Proof. Suppose not. Let $d' = \text{deg}_{C'}(E)$. Consider the filtration B induced on W by $[E \rightarrow E_{C'}]$. Since $\dim W = d + 2(1 - g) > d' + 2(1 - g) = w(B)$, B is a nontrivial filtration. Further,

$$\text{n.l.c. } w_B(m, C) \geq \text{n.l.c. } w_B(m, C') \geq 4d',$$

from Lemma 7.5. Thus

$$\frac{d}{d + 2(1 - g)} \cdot (d' + 2(1 - g)) \geq \frac{1}{4} \text{n.l.c. } w_B(m, C) \geq d'.$$

This contradicts $d' < d$.

Lemma 9.3. $H^1(C_{\text{red}}, \wedge^2 E) = 0$.

Proof. Suppose not. Lemma 9.1 shows there is a curve $C' \subseteq C_{\text{red}}$ with

$$h^0(C', \wedge^2 E) \leq \frac{1}{2} \text{deg}_{C'} E + 1.$$

Thus C' is not rational, and therefore Lemma 8.1 shows $\text{deg}_{C'}(E) \geq 20g$. On the other hand, E is generated by global sections, so we can find a nowhere vanishing section of E over C' :

$$(9.3.1) \quad 0 \rightarrow \mathcal{O}_{C'} \rightarrow E_{C'} \rightarrow (\wedge^2 E)_{C'} \rightarrow 0.$$

Hence

$$h^0(C, E) \leq \frac{\text{deg}_{C'}(E)}{2} + 2 \leq \text{deg}_{C'}(E) + 2 - 10g.$$

In particular,

$$h^0(C', E) < \deg_{C'}(E) + 2(1 - g),$$

which contradicts Lemma 9.2.

Lemma 9.4. $H^1(C_{\text{red}}, E) = 0$.

Proof. Suppose not. Then there is a nonzero map $\varphi: E \rightarrow \omega_{C_{\text{red}}}$. Using the techniques of the proof of Lemma 9.1, we can find a curve C' of C_{red} of genus g' and a map $\varphi: E \rightarrow \omega_{C'}$ which is nonzero on each component of C' . Note $g' \geq 2$, since otherwise E would have a trivial quotient. Then from (9.3.1),

$$h^0(C', E) \leq h^0(C', \wedge^2 E) + 1 \leq \deg_{C'}(E) + 1 - g' + 1,$$

since $H^1(C', \wedge^2 E) = 0$. We see $\deg_{C'}(E) \geq 20g$ from Lemma 8.1. Further $g' \leq 2g$, since otherwise

$$h^0(C', E) < \deg_{C'}(E) + 2(1 - g),$$

contradicting Lemma 9.2.

Now consider the filtration induced on W by $[E \rightarrow \omega_{C'}]$. We have $h^0(C', \omega_{C'}) = g'$, so $\sum r_i \leq g'$. We also have

$$\text{n.l.c. } w_B(m, C) \geq 2 \deg_{C'}(E),$$

from Lemma 7.4. So

$$4(2g) \geq 4g' \geq 4 \sum r_i \geq 2 \deg_{C'}(E) \geq 40g.$$

Hence we reach a contradiction.

Corollary 9.5. C is reduced and $W = H^0(C, E)$.

Proof. Consider \mathcal{G} , the ideal defining C_{red} in C . \mathcal{G} is supported at a finite number of points. We claim

$$(9.5.1) \quad W \cap H^0(C, \mathcal{G} \cdot E) \neq 0.$$

Let g' be the genus of C_{red} , and l be the length of \mathcal{G} . Then $g' = g + l$. Thus if $l > 0$, then

$$H^0(C_{\text{red}}, E) < \deg E + 2(1 - g) = \dim W,$$

since $H^1(C_{\text{red}}, E) = 0$. So (9.5.1) is established.

Now consider the filtration B induced on W by

$$\begin{pmatrix} E \cdot \mathcal{G} & E \\ 0 & 1 \end{pmatrix}.$$

Then $\sum r_i < \dim W$, but n.l.c. $w_B(m, C) = 4d$. We have again reached a contradiction.

Proposition 9.6. *C is semistable.*

Proof. Suppose $C = C' \cup C''$, where $C' \cap C''$ is a point P , and C'' is a chain of rational curves. The genus of C' is g , so

$$h^0(C', E) = \deg_{C'}(E) + 2(1 - g).$$

We have contradicted Lemma 9.2. So C is semistable.

10. Our purpose in this section is to establish some properties of E .

Proposition 10.1. *Let L be a quotient of E . Then $2 \deg_C L \geq \deg_C E$.*

Proof. Let $M = \ker(E \rightarrow L)$. Consider the filtration B :

$$\begin{pmatrix} M & E \\ 0 & 1 \end{pmatrix}.$$

It is easy to see B is destabilizing if $2 \deg L < \deg E$.

Now suppose $C' \subseteq C$ is a chain of rational curves $C_1 \cup \dots \cup C_l$, where the C_i are nonsingular rational, and C_i meets only C_{i-1} and C_{i+1} . We further suppose that $C'' = \overline{C - C'}$ is connected, and that C'' meets C_1 at one point P and C_l at one point Q , and meets no other C_i .

Lemma 10.2. $\deg_{C'}(E) \leq 2$.

Proof. Suppose not. The genus of C'' is $g - 1$. Consider the filtration B induced on W by $[E \rightarrow E_{C''}]$. First, notice that since $3 \leq d' = \deg_{C'} E$, and E is generated by global sections over C' , $H^0(C', E) > 4$. Hence the filtration B is nontrivial. We claim that

$$(10.2.1) \quad \text{n.l.c. } w_B(m, C') \geq 8.$$

Suppose (10.2.1) has been established. Let $d'' = d - d'$. Then $h^0(C'', E) = d'' + 2(2 - g)$, since C'' has genus $g - 1$. So

$$\frac{d}{d + 2(1 - g)} [d'' + 2(2 - g)] \geq d'' + 2.$$

After a short computation, we obtain $d' \leq 2$.

To establish (10.2.1), consider case one: $l = 1$. If we let $R = P + Q$, and apply (7.2.1) if $d' \geq 4$ and (7.2.2) if $d' = 3$, then we obtain (10.2.1). Next, consider case two: $d' = 3$. We claim that $H^0(C', \wedge^2 E(-2P - 2Q)) = 0$. Let s be such a nonzero section. We must have $\deg_{C_1} \wedge^2 E = 1$ or $\deg_{C_l} \wedge^2 E = 1$, since $d' = 3$. Say $\deg_{C_1} \wedge^2 E = 1$. Then s vanishes on C_1 , and therefore on $C_1 \cap C_2$. If $l = 2$, s vanishes twice at Q and once at $C_1 \cap C_2$, and so s vanishes. If $l = 3$, then $\deg_{C_3}(\wedge^2 E) = 1$. So s vanishes on C_3 also. But then s

vanishes on C_2 as well, since $\deg_{C_2} E = 1$. Hence $H^0(C', \wedge^2 E(-2P - 2Q)) = 0$. So the filtration induced by B on $\wedge^2 E_{C'}$ is dominated by

$$\begin{pmatrix} E(-P - Q) & E \\ & 1 & 2 \end{pmatrix}.$$

Applying Lemma 7.2, (10.2.1) holds, and $d' < 2$.

By applying cases one and two to subchains of C , we may assume that E does not have degree 3 on any subchain, and that $\deg_{C_i} E \leq 2$ for each i . It follows that the degree of E' on each C_i is two. But applying Lemma 7.2, we see

$$w_B(m, C_1) \geq 4, \quad w_B(m, C_l) \geq 4.$$

Then using Lemma 6.4, (10.2.1) holds, and $d' \leq 2$.

Now suppose the stable model C_s of C is an irreducible curve with a node N . Let \tilde{C}_0 be the normalization of C_s , and $d' = \deg \tilde{E}_0$.

Lemma 10.3. *Assume d to be odd. Let L be a quotient of \tilde{E}_0 . Then $2 \deg L \geq d - 1$ if $d = d'$, and \tilde{E}_0 is semistable if $d \neq d'$.*

Proof. Suppose for some $\delta \geq 0$

$$(10.3.1) \quad 2 \deg L \leq d - 2 - \delta.$$

Then

$$(10.3.2) \quad h^0(L) \leq \frac{1}{2}d + 1 - g + \frac{1}{2}\delta.$$

Indeed, if $h^1(L) = 0$, (10.3.2) follows from Riemann-Roch. If $h^1(L) \neq 0$, then $h^0(L) \leq g - 1$. But $d' \geq 20g$ (Lemma 8.1). So (10.3.2) follows in any case.

Now consider the weighted filtration B on W induced by $[\tilde{E} \rightarrow L]$. First, suppose $C = C_s$, and let $P, Q \in \tilde{C}_0$ be the points corresponding to N . Now \tilde{E}_P and \tilde{E}_Q are identified with E_N . Under this identification, $L_P \neq L_Q$ as quotients. Indeed, if $L_P = L_Q$, then L descends to a line bundle on C . This possibility is ruled out by Proposition 10.1. Thus if $M = \ker(\tilde{E}_0 \rightarrow L)$, then B is dominated by the filtration induced by

$$B' = \begin{pmatrix} M(-P - Q) & \tilde{E}_0 \\ & 0 & 1 \end{pmatrix}.$$

From Lemma 7.4 we see

$$\text{n.l.c. } w_B(m, C_0) \geq 2d + 2.$$

Combining these inequalities with n.l.c. $w_B(m, C) \leq 4dw(B)/n$, we obtain

$$(10.3.3) \quad \frac{d}{d + 2(1 - g)} \left(\frac{d}{2} + 1 - g \right) \geq \frac{1}{4}(2d + 2).$$

A short computation shows (10.3.3) is impossible.

Next suppose that $d \neq d'$ and that \tilde{E}_0 is not semistable. Since $d - d' \leq 2$ and d is odd, we may assume there is an L satisfying (10.3.1) with $\delta = 1$. Now letting $C' = \overline{C - C_0}$, we see

$$\text{n.l.c. } w_B(m, C') \geq 2, \quad \text{n.l.c. } w_B(m, C_0) \geq 2d'.$$

As above, this leads to

$$(10.3.4) \quad \frac{d}{d + 2(1 - g)} \left(\frac{d}{2} + 1 - g - \frac{1}{2} \right) \geq \frac{1}{4}(2d' + 2).$$

A short computation shows (10.3.4) cannot occur.

Thus we have established (1.3.1), (1.3.3) and (1.3.4).

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